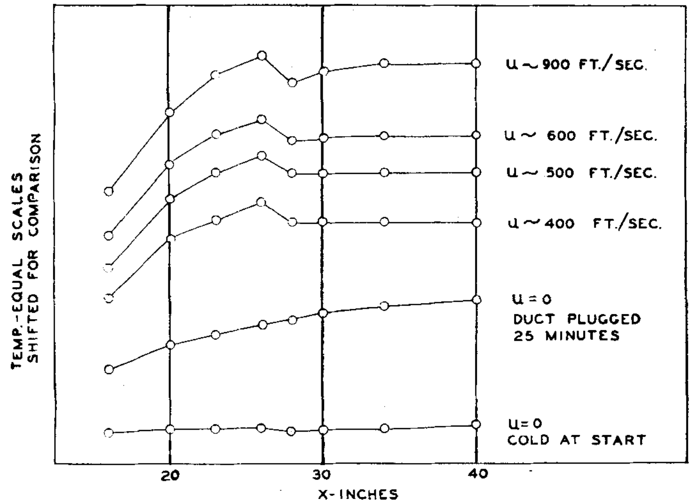

FIG. 15. VARIATION OF WALL TEMPERATURES WITH AIR RATE.

were computed at several points along the duct. The deviation of the measured values of mass flux from the mean value was small; the corresponding deviation in the case of heat flux was appreciably larger.

ACKNOWLEDGMENT

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NOTATION

C_f = impact tube coefficient
C_p = specific heat at constant pressure
c = momentum-spreading coefficient, empirical constant in Equation (19)
g_c = 32.2 ft. lb. (mass)/lb. (force) sec.^{2}
H = energy flux at any section. B.t.u./sec.
P = static pressure in duct
P_{atm} = static pressure in atmosphere
ΔP = pressure indicated by manometer, one end connected to a total head impact tube in a duct and the other open to the atmosphere
R = thermocouple recovery factor, Equation (1)
r = radial coordinate, distance from axis of duct
T = temperature
ΔT = temperature difference, Equation (8)
u = x-directed component of velocity
W = mass flux at any section, lb./sec.
x = axial coordinate, distance along axis of duct from nozzle
ρ = density of fluid

Subscripts

atm = atmosphere
f = free stream or static condition
i = induced stream conditions in plane of nozzle
in = indicated by thermocouple
n = upstream from flow nozzle
or = upstream at orifice
s = stagnation
w = condition in duct wall

LITERATURE CITED

1. Alexander, L. G., E. W. Comings, H. Grimmett, and E. W. White, *Chem. Eng. Progr. Symposium Series No. 10*, **50**, 93 (1954).
2. "Fluid Meters, Their Theory and Application," 4th ed., Am. Soc. Mech. Engrs. (1937).
3. Baron, T., and L. G. Alexander, *Chem. Eng. Progr.*, **47**, 181 (1951).
4. Danielson, R. D., M. S. Thesis, Univ. Illinois (1953).
5. Forstall, W., and A. H. Shapiro, *J. Appl. Mechanics*, **17**, 399 (1950).
6. Hinze, J. O., and B. G. van der Hegge Zignen, *Appl. Sci. Research*, A-1, 435 (1949).
7. Hottel, H. C., and A. Kalitinsky, *J. Appl. Mechanics*, 12, 25 (1945).
8. Landis, F., and A. H. Shapiro, Preprints of papers presented at the Heat Transfer and Fluid Mechanics Institute (1951).
9. Taylor, G. I., *Proc. Roy. Soc. (London)*, **135A**, 685 (1932).
10. Taylor, J. F., H. L. Grimmett, and E. W. Comings, *Chem. Eng. Progr.*, **47**, 175 (1951).

II. Theoretical Study of Turbulent Transport of Momentum, Energy, and Matter in Ducted Coaxial Streams

This paper concerns the kinetics of the processes that take place when a high-velocity jet of fluid mixes turbulently with a low-velocity, induced stream of the same fluid in a duct of uniform diameter. Semi-two-dimensional solutions of the equations of transport involving two empirical coefficients were obtained by application of Reichardt's hypothesis and three assumptions: (*a*) a negligible fraction of the flow entity (energy, mass, or momentum) is lost at the wall and the presence of the boundary layer may be ignored, (*b*) the static pressure is uniform over a section of the duct, and (*c*) the turbulence pattern is similar to

that in free jets except that the duct wall limits the growth of the scale of the turbulence. A mixing index, which is a measure of the degree to which the jet stream remains unmixed with the induced stream at any point, was defined and related to the geometry of the system and the flow parameters by means of the theoretical equation, and a method of evaluating the empirical coefficients for a typical case of momentum transport was described.

The physics of the exchange of energy, mass, and momentum between fluid streams is of theoretical and practical importance. Examples of application are jet pumps, Venturi atomizers, spray driers, combustion chambers, and gas burners.

Fluid-mixing processes fall into two general categories: the isobaric, as in free jets, and the nonisobaric, as in the jet ejector. An idealized example of the latter consists of a circular nozzle discharging an incompressible fluid into a cylindrical duct open at both ends to a second fluid. If the momentum losses can be estimated or neglected, the flow rates may be determined from jet- and induced-stream stagnation pressures and temperatures, flow areas, and discharge pressure by well known methods of thermodynamics.

Qualitatively the mixing of the streams is conceived to proceed as follows. The high-velocity jet stream imparts momentum to the fluid in the duct, causing it to flow toward the discharge. This lowers the pressure at the inlet, whereupon exterior secondary fluid begins to flow into the duct. Eventually a steady state is reached. In the induced stream the static pressure falls between the inlet and the plane of the jet nozzle, where the mixing process begins. The momentum, mass, and energy of the jet stream spread by turbulent convection into the secondary stream, and the static pressure rises toward the discharge pressure. If the mixing tube is long, so that mixing is nearly complete in, for example, one third of the length, then the static pressure must rise above the discharge pressure by an amount sufficient to overcome the resistance to flow of the remaining two thirds of length.

The purpose of this research was to set forth quantitatively the kinetics of the mixing process in such systems.

HISTORY

General. The literature on the mixing of fluid streams in a duct is too extensive to be reviewed in detail here. The greater part of the work deals with the over-all dynamics of jet ejectors. The thermodynamic theories appear to be satisfactory, and predictions of the optimum design of ejectors have been reasonably well verified by numerous experiments.

In contrast, theoretical and experimental studies of the kinetics of the mixing process are relatively few. Some investigations of the over-all performance included limited tests to ascertain that the lengths of the mixing sections were at least adequate. The conclusions have been of an approximate nature and are not all in agreement. It is generally thought that the length of the mixing section should be from four to eleven times the diameter, with seven diameters probably optimum (*8, 10, 12*). Opinions differ concerning the variation of this optimum length with entrance and exit geometry and pressures and with the velocities of the jet and induced streams (*8, 11, 12*). Of the experimental investigations of the mixing process, those of Forstall and Shapiro (*7*), Ledgett (*11*), Ferguson (*5*), Reid (*14*), and Viktoren (*15*) are significant. These authors measured impact pressures at various points in the zone of mixing. Ledgett and Viktoren also measured static pressures throughout this zone, and their results agree qualitatively with the description given in this introduction.

Flugel's (*6*) analysis involves the crude assumption that at any cross section individual average velocities may be assigned to the high-velocity core and the low-velocity flow surrounding it. The drag force between the two streams is then equated to the product of the square of the difference between their velocities and an empirical drag coefficient. His equations satisfy the momentum balance but have little relation to the mechanics of the mixing process.

Viktoren (*15*) obtained a solution that approximates the flow pattern in the limited region of mixing where the distance from the nozzle is sufficiently great that the jet may be regarded as a point source of mass and momentum and yet sufficiently close to the nozzle that the spreading of the jet is not affected by the presence of the wall. His assumptions were that (*a*) the pressure is uniform throughout the mixing tube, (*b*) the mixing processes in cross sections at various distances from the nozzle are geometrically and mechanically similar, and (*c*) the Prandtl mixing length l_P is uniform over a cross section and, moreover, proportional to the width of the mixing zone. These assumptions seriously limit the value of his solution. In some cases, as with jet pumps, the pressure rise in the mixing tube is the most important effect. It is clear that the flow profiles and mixing processes cannot be similar, as supposed by Viktoren, except in a limited region which may not exist at all unless the jet nozzle is very small compared with the duct. That is, for a point-source jet the successive flow profiles will be similar as in a free jet until the spread of jet momentum is inhibited by the wall. But it is precisely a knowledge of the effect of the wall on the mixing process that is desired. A solution limited to the region in a duct where the flow approximates that in a free jet gives no information concerning the length of duct required to mix the two streams to any given degree.

Reichardt's Hypothesis. Herman Reichardt, of Göttingen, has proposed a formula for the turbulent shearing stress (*13*). In previous publications from this laboratory (*1*) the singular success of Reichardt's hypothesis and the principle of superposition deduced from it by Baron and Alexander (*3*) in correlating measurements in free jets was demonstrated. The flow pattern in the region adjacent to the nozzle was predicted with reasonable success. The treatment also was applied in the region of confluence of twin parallel jets and in the region of flow of a free jet parallel to a plane surface (*1*). Reichardt's hypothesis has been generalized to include the transport of heat and mass in a free turbulent jet (*3*).

The hypothesis relates the turbulent shearing stress to the gradient of the momentum flux density.

$$\rho\,\overline{uv} = -\Lambda\frac{\delta\rho\overline{u^2}}{\delta r} \qquad (1)$$

where Λ is a scale factor somewhat analogous to Prandtl's mixing length and is to be related to the boundary conditions. Setting Λ equal to $\frac{c^2x}{2}$ leads to the well known distribution of momentum flux in a free jet (*1*) where c is reasonably invariant. There appear to be some second-order variations in c with changes in boundary conditions, but these have not been delimited (*1*).

KINETICS OF MIXING IN THE DUCTED JET

General Solution. When viscous and body forces may be neglected, the equation of motion for axially symmetrical turbulent flow becomes (*1*)

$$\frac{\delta\overline{P}}{\delta x} + \frac{\delta\rho\overline{u^2}}{\delta x} + \frac{1}{r}\frac{\delta}{\delta r}\left[r\rho\overline{uv}\right] = 0 \quad (2)$$

where x and r are axial and radial coodinates, u and v are axial and radial components of velocity, P is the static pressure, and ρ is the density. The bar denotes time average values:

$$\overline{P} = \lim_{\theta \to \infty} \frac{1}{\theta} \int_{\theta_o}^{\theta_o + \theta} P \, d\theta \qquad (3)$$

where θ is the time of observation.

Application of Reichardt's hypothesis with certain approximations leads to a semi-two-dimensional solution of the axial flow in a tube with mixing induced by a jet. Substitution of Equation (1) into Equation (2) gives

$$\frac{\delta \overline{P}}{\delta x} + \frac{\delta \overline{\rho u^2}}{\delta x} - \frac{1}{r} \frac{\delta}{\delta r} \left[r\Lambda \frac{\delta \overline{\rho u^2}}{\delta r} \right] = 0 \qquad (4)$$

The sum $\overline{P} + \overline{\rho u^2}$ is recognized as the total momentum. If it is assumed that the static pressure is uniform over a cross section, then

$$\frac{\delta \overline{P}}{\delta r} \cong 0 \qquad (5)$$

Equation (5) is almost certainly untrue in detail. Measurements of the static-pressure distribution reported in the literature are meager and equivocal. Viktoren (*15*), studying a system employing water for both jet and induced fluids, found large apparent variations across the duct near the jet nozzle and concluded that static-pressure measurements in such regions are not reliable. Ledgett (*11*), however, using air, studied a system very similar to that treated here and found the variations to be less than 10% of the dynamic head in the same region. Hence Equation (5) is probably a reasonably good first approximation. Equation (4) may be written

$$\frac{\delta}{\delta x} (\overline{P} + \overline{\rho u^2}) - \frac{1}{r} \frac{\delta}{\delta r} \left[r\Lambda \frac{\delta}{\delta r} (\overline{P} + \overline{\rho u^2}) \right] = 0 \qquad (6)$$

This equation is linear in $\overline{P} + \overline{\rho u^2}$ and a solution may be obtained in orthogonal functions.

To simplify the boundary conditions, a variable, *M*, is introduced such that

$$M = \frac{m - m_\infty}{m_{j,\, avg} - m_{i,\, avg}} \qquad (7)$$

and

$$m = \overline{P} + \overline{\rho u^2} \qquad (8)$$

where the subscript *j* denotes the initial jet stream; *i*, the initial induced stream; ∞, the final, completely mixed stream. In the definition of m_∞, it is assumed that friction drag at the walls may be neglected and that the final distribution of momentum flux is uniform over a cross section. To simplify the mathematics, only the case where m_j and m_i are uniformly distributed in the plane of the nozzle discharge will be treated; however, any arbitrary distribution can be treated. The variable *M* satisfies Equation (6), which may be written

$$\frac{\delta M}{\delta x} - \frac{4}{r_* D^2} \frac{\delta}{\delta r_*} \left[r_* \Lambda \frac{\delta M}{\delta r_*} \right] = 0 \qquad (9)$$

where *D* is the diameter of the duct, and r_* is a normalized radial coordinate such that

$$r_* = \frac{2r}{D} \qquad (10)$$

If it is assumed that λ is a function of *x* only, a solution is readily obtained by separating the variables.

$$M = \left[AJ_o(Kr_*) + BY_o(Kr_*) \right] exp - \int_o^x \frac{4K^2\Lambda}{D^2} dx \qquad (11)$$

Differentiation with respect to r_* gives

$$\frac{\delta M}{\delta r_*} = \left[AKJ_1(Kr_*) + BKY_1(Kr_*) \right] exp - \int_o^x \frac{4K^2\Lambda}{D^2} dx \qquad (12)$$

The boundary condition that

$$\frac{\delta M}{\delta r_*} = 0;\; r_* = 0 \qquad (13)$$

requires that both terms in the bracket vanish on the axis. Then

$$J_1(0) = 0 \qquad (14)$$

as required, but

$$Y_1(0) = -\infty \qquad (15)$$

hence

$$B = 0 \qquad (16)$$

Neglecting friction requires that no momentum be lost at the wall; hence

$$\frac{\delta M}{\delta r_*} \cong 0;\; r_* = 1 \qquad (17)$$

This condition is satisfied by Equation (12) provided *K* is a root of J_1.

The particular solutions may now be combined:

$$M = \sum_1^\infty A_n J_o(a_n r_*) \; exp - \int_o^x \frac{4a_n^2\Lambda}{D^2} ax \qquad (18)$$

where a_n is the *n*-th root of J_1.

The quantity Λ is probably related to the scale and intensity of the turbulence. In the vicinity of the nozzle the turbulence pattern should be similar to that in the corresponding region in a free jet. Far from the nozzle, the duct wall modifies the flow in at least two ways: it (*a*) confines the flow of momentum and (*b*) modifies the turbulence pattern in that the turbulence in the vicinity of the wall is damped. This latter phenomenon is associated with the transfer of momentum to the wall by a boundary layer. Since friction at the wall has been neglected throughout, the only mathematically expressible property of the wall left is that of being a geometric boundary, and its action is represented by Equation (17). The corollary assumption regarding Λ, then, is that it is constant across a section and is a function only of *x*.

Implicit in these assumptions is the prediction that because of the high level of turbulence generated by the jet, the velocity profile will become substantially uniform over the duct before the boundary layer has time to develop significantly.

In fully developed turbulent flow in a pipe the turbulence level is only about 3%. In free jets the level reaches 20% and higher (*4*). With a jet mixing in a duct, the turbulence level will decrease from that of a free jet to that of normal pipe flow. On the other hand, the scale of turbulence, as measured by Λ, cannot grow indefinitely in a duct, as it does in a free jet, and the relation between Λ and x must take this into account. A purely arbitrary function that reduces to the free-jet relation in the vicinity of the nozzle, but which imposes on Λ an upper limit which is a function of the duct diameter, is

$$\Lambda = \frac{c^2 x}{2\left[1 + \frac{bx}{D}\right]} \tag{19}$$

where b and c are empirical coefficients. Substitution of this into Equation (18) followed by integration gives

$$M = \sum_1^\infty A_n J_o(a_n r_*) \exp\left\{ - \frac{2a_n^2 C^2}{b^2}\left[\frac{bx}{D} - \ln\left(1 + \frac{bx}{D}\right)\right]\right\} \tag{20}$$

For the case where b is zero, this reduces to

$$M = \sum_1^\infty A_n J_o(a_n r_*) \exp\left[- a_n^2 \left(\frac{cx^2}{D}\right)\right] \tag{21}$$

The remaining boundary conditions are

$$M \longrightarrow 0 \text{ as } x \longrightarrow \infty$$

and

$$M_o = f(r_*)$$

where the subscript o denotes the plane of the jet nozzle. The first condition results from the fact that m approaches m_∞ and is satisfied by the exponential term, which tends toward zero as x increases without limit. The second condition is imposed on the A_n by use of the Bessel series.

$$A_n = \frac{2}{J_o^2(a_n)} \int_o^1 r_* M_o J_o(a_n r_*)\, dr_* \tag{22}$$

Although any arbitrary initial profile of total momentum can be imposed on the A_n, the only case developed here will be that where the jet velocity is uniform over the plane of the nozzle and the velocity of the induced stream is uniform in the same plane. It follows that

$$M_j = \frac{m_j - m_\infty}{m_j - m_i}; 0 < r_* < \frac{d}{D} \tag{23}$$

where d is the diameter of the jet nozzle and

$$M_i = \frac{m_i - m_\infty}{m_j - m_i}; \frac{d}{D} < r_* < 1 \tag{24}$$

Substitution in Equation (22) and integration results in

$$A_n = \frac{2\frac{d}{D} J_1\left(a_n \frac{d}{D}\right)}{a_n J_o^2(a_n)} \tag{25}$$

Thus the definition of M chosen in Equation (18) renders the A_n dependent only on the ratio d/D and independent of the initial velocities for the case under consideration.

The quantity m_∞ may be evaluated from the fact that the total momentum, with friction neglected, is conserved. Thus

$$\int_o^1 M 2\pi r_*\, dr_* = \text{const} (=0) \tag{26}$$

Since M tends toward zero with increasing x, the constant must be zero. Substitution of Equations (23) and (24) into Equation (26) followed by integration and solution for m_∞ gives

$$m_\infty = m_j \left(\frac{d}{D}\right)^2 + m_i\left[1 - \left(\frac{d}{D}\right)^2\right] \tag{27}$$

The principal defect of the treatment given here is that a knowledge of M does not immediately yield information concerning the distribution of the static pressure, $\overline{P}$, nor of the flow momentum, $\overline{\rho u^2}$. A knowledge of the distribution of the turbulence intensity is required. It is thus desirable to expand the flow variables in mean and fluctuating components, as follows:

$$u = \bar{u} + u'$$
$$v = \bar{v} + v' \tag{28}$$
$$P = \overline{P} + P'$$

where the mean value is denoted by the bar and the primed quantity is the fluctuating component. By definition

$$\overline{u'} = 0$$
$$\overline{v'} = 0 \tag{29}$$
$$\overline{P'} = 0$$

Combining Equation (8) and (28) gives

$$\rho\bar{u} = \sqrt{\rho\left(m - \overline{P} - \rho\overline{u'^2}\right)} \tag{30}$$

whence the total flux of mass is obtained by integrating $\overline{\rho u}$ over a cross section.

$$\int_o^1 \frac{D^2}{2}\sqrt{\rho\left(m - \overline{P} - \rho\overline{u'^2}\right)}\,\pi r_* dr_* = \text{mass flux} \tag{31}$$

The distribution of m may be obtained from that of M. To evaluate $\overline{P}$ and obtain the distribution of $\overline{\rho u^2}$ at any cross section, it is necessary to choose values of $\overline{P}$, perform the integration numerically, and compare the result with the integral mass flux until agreement is obtained. However, there is no satisfactory way of estimating $\overline{u'^2}$ and experimental data are nonexistent. Possibly the best course at present is to assign to $\overline{u'^2}$ an average value, based on Corrsin's free-jet data (4), of 0.04 $\overline{u^2}$.

The tedious procedure outlined above is neither necessary nor desirable for comparing Equations (20) and (21) with measured values. If the turbulence components are to be neglected or approximate values used, it is better to do so in connection with the quantity m.

In experimental work the flow is conveniently explored with a total-head impact tube connected to a manometer having one arm open to the atmosphere. Unfortunately impact-tube measurements do not immediately yield a knowledge of $\overline{\rho u^2}$ (1). So far as turbulence is concerned, impact tubes should respond in (locally) incompressible flow according to the equation

$$\Delta P = \overline{P} + \frac{\rho \overline{u^2} + \rho(\overline{v'^2} + \overline{w'^2})}{2g_c C_f^2} - P_{atm}$$

To obtain $\rho\overline{u^2}$, a knowledge of $\overline{v'^2}$ and $\overline{w'^2}$ is required. Since this is, in general, not available, these terms must be neglected, and approximately

$$m \cong 2g_c C_f^2 (\Delta P + P_{atm} - \overline{P}) + g_c \overline{P} \quad (32)$$

where ΔP is the manometer pressure reading in consistent units, C_f is an impact-tube coefficient, and P_{atm} is the atmospheric pressure.

The Mixing Index. There is as yet no way of predicting the values of the constants b and c, in Equation (19). In free jets discharging into a stagnant atmosphere, c ranges from 0.07 to 0.08 with a small variation with position in the jet, nozzle geometry, and discharge velocity(*1*). Calculations based on the data of Forstall and Shapiro(*7*) for a free jet discharging into a moving secondary stream give values of c down to 0.01 as the velocity of the secondary stream approaches the jet velocity(*9*).

Point-by-point application of Equations (20) and (21) is not a satisfactory means of evaluating b and c from experimental data. It is preferable that these be determined from a plot of the axial distribution of some function integral in r. Linear plots are to be desired but are not always obtainable. If wall friction is neglected, the total momentum flux is conserved, and the axial distribution of the left-hand member of Equation (26) gives no information concerning c and b. However, the integral of the square of the momentum flux density is not conserved and may be used to define a mixing index in terms of the deviation of the square of the momentum-flux density from the average value; thus

$$\gamma = \int_0^1 (M - M_{avg})^2 \, 2\pi r_* dr_* \quad (33)$$

where

$$M_{avg} = \int_0^1 M d\,(r_*^2) \quad (34)$$

But the latter integral has already been shown to be zero. Hence M_{avg} is zero. Substitution now from Equations (20) and (25) into Equation (33) gives

$$\gamma = \int_0^1 \left\{ \sum_1^\infty \frac{2\frac{d}{D} J_1\left(a_n \frac{d}{D}\right)}{a_n J_o^2(a_n)} J_o (a_n r_*) \exp - \frac{2a_n^2 c^2}{b^2} \left[\left(\frac{bx}{D}\right) - \ln\left(1 + \frac{bx}{D}\right) \right] \right\}^2 2\pi r_* dr_* \quad (35)$$

From the orthogonality of the series,

$$\int_0^1 r_* J_o(a_k r_*) J_o(a_n r_*) dr_* = 0 \quad (36)$$

for $k \neq n$, whereas

$$\int_0^1 r_* J_o^2 (a_n r_*) dr_* = \frac{1}{2}\left[J_o^2(a_n) + J_1^2(a_n) \right] \quad (37)$$

Hence, integration of Equation (35) leads to the equation

$$\gamma = 4\pi \left(\frac{d}{D}\right)^2 \sum_1^\infty \frac{J_1^2\left(a_n \frac{d}{D}\right)}{a_n^2 J_o^2(a_n)} \exp - \frac{4 a_n^2 c^2}{b^2} \left[\left(\frac{bx}{D}\right) - \ln\left(1 + \frac{bx}{D}\right) \right] \quad (38)$$

which converges rapidly except for small values of x. For the case where b is zero, Equation (38) reduces to

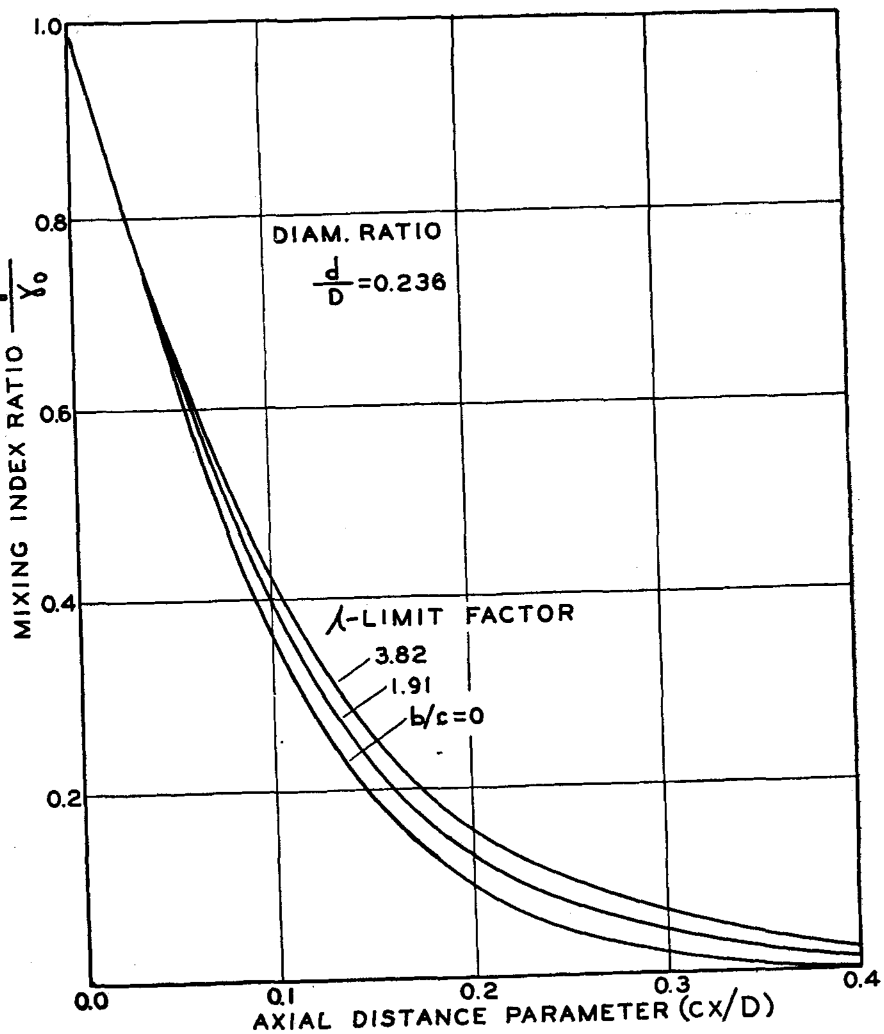

FIG. 1.

$$\gamma = 4\pi\left(\frac{d}{D}\right)^2 \sum_{1}^{\infty} \frac{J_1^2(a_n \frac{d}{D})}{a_n^2 J_o^2(a_n)} exp - 2a_n^2\left(\frac{cx}{D}\right)^2 \quad (39)$$

TABLE 1.—THE MIXING INDEX

Calculated from Equations (38) and (39) with $d/D = 0.23\gamma$

Axial distance parameter, cx/D	Λ-Limit factor, b/c 0	1.91	3.81
	Mixing index, γ		
0.00	0.1650	0.1650	0.1650
0.02	0.1408	0.1408	0.1408
0.04	0.1155	0.1200	0.1205
0.06	0.0975	0.0996	0.1018
0.08	0.0771	0.0779	0.0837
0.10	0.0606	0.0649	0.0683
0.15	0.0313	0.0371	0.0413
0.20	0.0163	0.0215	0.0258
0.30	0.0035	0.0076	0.0109
0.40	0.0005	0.0020	0.0045

Inspection of Equation (38) shows that γ approaches zero as x increases and that the initial value, γ_o, is dependent only on the diameter ratio, d/D. It may be evaluated by substitution from Equations (23) and (24) into Equation (33) followed by integration, whence

$$\gamma_o = \pi\left(\frac{d}{D}\right)^2\left[1 - \left(\frac{d}{D}\right)^2\right] \quad (40)$$

The mixing index, γ, may also be evaluated from experimental data by graphical integration of Equation (33).

Values of the mixing index have been calculated. In order to generalize the calculations, they were carried out by assigning various values to an axial distance parameter cx/D and to a Λ-limit factor, b/c. The results are listed in Table 1 and are shown graphically in Figure 1 for a diameter ratio, d/D, of 0.236.

For given values of x, b, and c, the corresponding value may be found readily by interpolation. The values of b/c chosen correspond to the condition that if c is 0.075, then Λ will have increased in seven duct diameters to the fractions 0, 1/2, and 2/3 of its maximum value respectively. It is seen that limiting the growth of Λ does not affect the decay of γ greatly except at small values of γ/γ_o.

Experimental data will show some loss of momentum through friction at the wall. This nonconservation of the experimentally determined flux of total momentum presents a difficulty in calculating the mixing index, γ, from the data. If m_∞ is calculated from Equation (27), Equation (26) is not borne out by the experimental measurements. It is important that the data satisfy both Equations (26) and (27). Accordingly, a variable $m_{\infty,x}$ was defined which is to be evaluated at each cross section:

$$m_{\infty, x} = \int_o^1 m dr_*^2 \quad (41)$$

That is, m_∞ at x is taken as the average value of m over the cross section at x.

The spreading coefficient is evaluated by assigning to it the value which gives the best fit between Equation (39) (if b is zero), normalized with respect to γ_o, and the experimental data points calculated from Equation (33), also normalized. The method of least squares may be applied conveniently. Let Δ be the deviation between theoretical and experimental values of γ/γ_o for a value of c that gives a reasonably good visual fit:

$$\Delta = \left(\frac{\gamma}{\gamma_o}\right)_e - \left(\frac{\gamma}{\gamma_o}\right)_t \quad (42)$$

From statistical theory, the most probable value of c is that which minimizes the sum of the squares of the deviations for all the data:

$$\frac{\delta(\Sigma\Delta^2)}{\delta c} = 0 \quad (43)$$

If the probable value, c_P, does not differ greatly from the value chosen visually, c_V, derivatives of orders higher than the first may be neglected, and in the interval $c_P - c_V$,

$$\Delta = \Delta_V + \frac{\delta\Delta}{\delta c}(c - c_V) \quad (44)$$

where Δ_V is the deviation obtained by visual matching. Substitution in Equation (43) followed by solution for c_P gives

$$c_P = c_V - \frac{\sum\left(\Delta_V \frac{\delta\Delta}{\delta c}\right)}{\sum\left(\frac{\delta\Delta}{\delta c}\right)^2} \quad (45)$$

But by Equation (42)

$$\frac{\delta\Delta}{\delta c} = -\frac{1}{c}\left(\frac{cx}{D}\right)\frac{\delta\left(\frac{\gamma}{\gamma_o}\right)_t}{\delta\left(\frac{cx}{D}\right)} \quad (46)$$

The differential is readily evaluated by differentiating Equation (39) after normalizing; whence

$$\frac{\delta\Delta}{\delta c} = \frac{\frac{4}{c}\left(\frac{cx}{D}\right)\sum_{1}^{\infty}\frac{J_1^2(a_n\frac{d}{D})}{J_o^2(a_n)} exp[-2a_n^2(\frac{cx}{D})^2]}{\sum_{1}^{\infty}\frac{J_1^2(a_n\frac{d}{D})}{a_n^2 J_o^2(a_n)}} \quad (47)$$

Now let there be defined a function $\varphi\left(\frac{cx}{D}\right)$ such that

$$\varphi\left(\frac{cx}{D}\right) = c\frac{\delta\Delta}{\delta c} \quad (48)$$

Substitution of Equations (46) and (48) into (45) gives

$$c_P \cong c_V\left[1 - \frac{\Sigma(\varphi\Delta_V)}{\Sigma(\varphi^2)}\right] \quad (49)$$

Values of φ for various selected values of (cx/D) are listed in Table 2.

TABLE 2.—THE FUNCTION $\varphi\left(\frac{cx}{D}\right)$

For evaluating c from the gamma correlation

(cx/D)	φ
0.00	0.000
0.02	0.152
0.04	0.297
0.06	0.407
0.08	0.468
0.10	0.484
0.15	0.417
0.20	0.232
0.25	0.155
0.30	0.093
0.40	0.021

This method of evaluating c weights heavily those data corresponding to middle values of cx/D. These are the most accurate data because near the nozzle turbulence interferes, and far down the duct, where γ is small, the boundary layer contributes significantly to the shape of the profile.

TRANSPORT OF ENERGY IN A DUCTED JET

If molecular transport and generation terms may be neglected, the equation for the turbulent transport of energy may be put in the form(*1*)

$$\nabla \cdot \rho C_p \overline{VT_s} = 0 \qquad (50)$$

which, in cylindrical coordinates for axially symmetrical flow, takes the form

$$\frac{\delta}{\delta x}(\rho C_p \overline{T_s u}) + \frac{1}{r}\frac{\delta}{\delta r}(r\rho C_p \overline{T_s v}) = 0 \qquad (51)$$

Here T_s denotes the total or stagnation temperature:

$$\rho C_p T_s = \rho C_p T_f + \frac{J\rho \overline{V^2}}{2g} \qquad (52)$$

where T_f is the free-stream temperature. Substituting and regrouping give

$$\delta\frac{(\rho C_p \overline{T_f u})}{\delta x} + \frac{1}{r}\frac{\delta}{\delta r}(r\rho C_p \overline{T_f v}) + \frac{\delta}{\delta x}\left(\frac{J\rho \overline{V^2 u}}{2g_c}\right) + \frac{1}{r}\frac{\delta}{\delta r}\left(\frac{rJ\rho \overline{V^2 v}}{2g_c}\right) = 0 \qquad (53)$$

Assume now that similarity exists between the transport of heat and momentum in free turbulence. The equation analogous to Reichardt's formula for shearing stress is(*1*)

$$\overline{T_f v} = -\Lambda_T \frac{\delta \overline{T_f u}}{\delta r} \qquad (54)$$

1. *Case of Negligible Kinetic Energy*

For this case the last two terms in Equation (53) drop out, and application of Equation (54) gives

$$\frac{\delta(\rho C_p \overline{T_f u})}{\delta x} - \frac{\Lambda_T}{r}\frac{\delta}{\delta r}\left(r\frac{\delta \rho C_p \overline{T_f u}}{\delta r}\right) = 0 \qquad (55)$$

which is linear in $\overline{T_f u}$. If one defines

$$m_T = \rho C_p \overline{T_f u} \qquad (56)$$

and substitutes Equations (7) and (10) into Equation (55), it becomes identical with Equation (9) except for subscripts. Therefore Equation (18) is a solution of Equation (55), and a mixing index for heat transport may be defined in the same way as that for momentum, by Equation (38) or (39).

2. *Case of Appreciable Kinetic Energy*

For Equation (53) to be linearized completely, it is necessary that the lateral transport of kinetic energy be governed by an equation similar to Equation (54) with the same value of Λ_T. But this law must also be consistent with the law governing the lateral transport of momentum. The approximations involved in assuming such a law for transport of kinetic energy to be valid will be investigated.

Expanding the velocity in mean and fluctuating components,

$$\overline{V^2 v} = 2\bar{u}\,\overline{u'v'} = 2\bar{u}\,\overline{uv} \qquad (57)$$

for axially symmetrical flow where $\bar{v} = \bar{w} = 0$.
Application of Equation (1) now gives

$$\overline{V^2 v} = -2\Lambda\, \bar{u}\frac{\delta \overline{u}^2}{\delta r} \qquad (58)$$

On the other hand,

$$\overline{V^2 u} = \bar{u}^3 + \bar{u}\,(3\overline{u'^2} + \overline{v'^2} + \overline{w'^2}) \qquad (59)$$

Now assuming that, to a first degree of approximation,

$$\overline{u'^2} \cong \overline{v'^2} \cong \overline{w'^2} \qquad (60)$$

whence

$$\overline{V^2 u} = \bar{u}^3 + 5\,\bar{u}\,\overline{u'^2} \qquad (61)$$

Differentiation with respect to r gives

$$\frac{\delta \overline{V^2 u}}{\delta r} = \frac{3}{2}\bar{u}\frac{\delta \bar{u}^2}{\delta r} + 5\bar{u}\frac{\delta \overline{u'^2}}{\delta r} + 5\overline{u'^2}\frac{\delta \bar{u}}{\delta r} \qquad (62)$$

Recombining terms gives

$$\frac{\delta \overline{V^2 u}}{\delta r} = \frac{3}{2}\bar{u}\frac{\delta \bar{u}^2}{\delta r} + \frac{7}{2}\bar{u}\frac{\delta \overline{u'^2}}{\delta r} + 5\overline{u'^2}\frac{\delta \bar{u}}{\delta r} \qquad (63)$$

If the last two terms may be neglected (they are small throughout most of the flow field of a free jet),

$$\frac{\delta \overline{V^2 u}}{\delta r} \simeq \frac{3}{2}\bar{u}\frac{\delta \overline{u}^2}{\delta r} \qquad (64)$$

The equation for kinetic energy transfer analogous to Equation (54) is

$$\overline{V^2 v} = -\Lambda_E \frac{\delta \overline{V^2 u}}{\delta r} \qquad (65)$$

Combining Equations (58), (64), and (65) and solving for Λ_E gives

$$\Lambda_E \cong \frac{4}{3}\Lambda \qquad (66)$$

It has been observed repeatedly that heat is dissipated more rapidly than momentum in turbulent flow, and the ratio of the relative rate factors has been given as approximately 4/3 in free jet(*8*). Therefore

$$\Lambda_T \cong \frac{4}{3}\Lambda$$

and consequently

$$\Lambda_E \cong \Lambda_T \qquad (67)$$

When this result together with Equations (52) and (54) is applied to Equation (53), there results

$$\frac{\delta}{\delta x}(\rho C_p \overline{T_s u}) - \frac{\Lambda_T}{r}\frac{\delta}{\delta r}\left[r\frac{\delta}{\delta r}(\rho C_p \overline{T_s u})\right] = 0 \qquad (68)$$

which is linear in $\overline{T_s u}$. By defining

$$m_s = \rho C_p \overline{T_s u} \qquad (69)$$

and using Equations (7) and (10), Equation (68) becomes identical with Equation (9) except for subscripts. Equation (18) is therefore a solution of Equation (68), and Equation (38) or (39) may be applied to the correlation of the stagnation temperature.

TRANSPORT OF MASS IN A DUCTED JET

For the transport of mass when there are no sources or sinks present and chemical interconversion is negligible:

$$\frac{\delta \rho \overline{gu}}{\delta x} + \frac{1}{r}\frac{\delta}{\delta r}\left[r\, \rho \overline{gr} \right] = 0 \quad (70)$$

where g is the mass fraction of jet-stream fluid in the mixture. If similarity to momentum transport exists, then

$$\overline{gv} = -\Lambda_g \frac{\delta \overline{gu}}{\delta r} \quad (71)$$

when

$$m_g = \rho \overline{gu}$$

Equation (70) is transformed into Equation (9), and the solution follows as before.

Of course, in order to solve Equation (21) for the distribution of energy or mass, the velocity profile must be known. A difficulty enters here in that the products $\overline{Tu}$ and $\overline{gu}$ involve cross products of the fluctuating components:

$$\overline{Tu} = \overline{T}\overline{u} + \overline{T'u'}$$

$$\overline{gu} = \overline{g}\,\overline{u} + \overline{g'u'}$$

$$\overline{u^2} = \overline{u}^2 + \overline{u'^2}$$

Presumably $\overline{T}$ can be measured with a thermocouple and $\overline{g}$ by chemical analysis, but the situation is equivocal. The mean products of the fluctuating components can be measured, in theory, with a suitable type of hot-wire anemometer (*4*), but such measurements are not easy. There is evidence (*1*) that neglect of these fluctuating components introduces errors of 30% or more into the correlations of energy and mass transport in free jets.

SUMMARY

It has been shown that (*a*) if Reichardt's formula applies to the turbulent transport of energy, mass, and momentum in ducted jets, (*b*) if molecular transport and boundary layer formation may be neglected, and (*c*) if radial pressure gradients and heat losses at the wall may be neglected, then the transport equations are linearized in the corresponding fluxes, and solutions may be obtained in orthogonal functions. This was done. In order to permit evaluation of the empirical constants appearing in the solutions, a function γ, the mixing index, was devised. It is a measure of the degree to which the two streams remain unmixed at any point, and it approaches zero as the mixing proceeds. Methods of correlating experimental data in terms of γ were given.

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NOTATION

- A = cross-sectional area of mixing duct
- A_n = arbitrary coefficient of nth term in a series of J_o
- a_n = nth root of J_1
- B_n = arbitrary coefficient of nth term in series of Y_o
- b = empirical constant in Equation (19)—a measure of the action of the duct wall in inhibiting the growth of the scale of the turbulence
- C_f = impact-tube coefficient
- C_p = specific heat at constant pressure
- c = momentum spreading coefficient, empirical constant in Equation (19)
- D = diameter of mixing duct
- d = diameter of jet nozzle
- g = fraction by weight of jet fluid in mixture
- J_o = zero-order Bessel function of the first kind
- J_1 = first-order Bessel function of the first kind
- K = constant of integration in Equation (11); the boundary conditions require that K be a root of J_1
- M = variable defined by Equation (7); M is proportional to the flux density of total momentum, energy, or mass
- m = flux density of total momentum, energy, or mass, Equation (8)
- P = static pressure in duct
- P_{atm} = static pressure in atmosphere
- ΔP = pressure indicated by manometer one end of which is connected to a total-head impact tube in a duct and the other open to the atmosphere
- r = radial coordinate, distance from axis of duct
- r_* = normalized radial coordinate, Equation (10)
- T = temperature
- u = x-directed component of velocity
- v = r-directed component of velocity
- x = axial coordinate, distance along axis of duct from nozzle
- Y_o = zero-order Bessel function of second kind
- Y_1 = first-order Bessel function of second kind

Greek

- γ = mixing index, Equation (33)
- Λ = proportionality function in Reichardt's shearing-stress equation, function of x and r, Equation (1)
- ρ = density of fluid
- θ = time

Subscripts

- avg = average
- atm = atmosphere
- E = transport of energy
- f = free stream
- g = transport of mass
- i = induced stream conditions in plane of nozzle
- j = jet-stream conditions in plane of nozzle
- k = kth term in series of J_o
- s = stagnation
- T = transport of heat
- x = conditions in cross section of flow at distance x from nozzle
- o = conditions in plane of jet nozzle
- ∞ = condition that would be obtained by complete mixing of the flow at any section with no further loss of momentum or energy

LITERATURE CITED

1. Alexander, L. G., T. Baron, and E. W. Comings, *Univ. Illinois, Eng. Expt. Sta. Bull.* 405 (1952)
2. Alexander, L. G., E. W. Comings, H. L. Grimmett, and E. A. White, *Chem. Eng. Progr., Symposium Series No. 10,* **50,** 93 (1954).
3. Baron, T., and L. G. Alexander, *Chem. Eng. Progr.,* **47,** 181 (1951).
4. Corrsin, S., and M. S. Uberoi, *Natl. Advisory Comm. Aeronaut. Tech. Note* 1865 (1949).
5. Ferguson, C. K., 1949 Fluid Mechanics and Heat Transfer Institute, Berkeley, Calif. (1949).

6. Flugel, G., *Forsch. Gebiete Ingenieurw., Forschungsheft*, 395 (1939); *Natl. Advisory Comm. Aeronaut. Tech. Mem* 982 (1941).
7. Forstall, W., and A. H. Shapiro, *J. App. Mech.*, 17, 399 (1950).
8. Keenan, J. H., et al., *J. App. Mech.*, 17, 299 (1950).
9. Kivnick, A., *Eng. Expt. Sta. Univ. Illinois Tech. Rept.* 2, DA-18-064-CML-445 (1952).
10. Kroll, H. E., *Chem. Eng. Progr.*, 43, 21 (1947).
11. Ledgett, L. A., Preprint, Am. Soc. Mech. Engrs., Summer Meeting (1934), George Reproduction Co. (1934).
12. McElroy, G. E., *U. S. Bur. Mines Tech. Paper* 678 (1945).
13. Reichardt, H., *Z. Math. u. Mech.*, 21, 257 (1941).
14. Reid, E. G., *Natl Advisory Comm. Aero Tech. Note* 1949 (1949).
15. Viktoren, K., *Forsch. Gebiete Ingenieurw.* 12, 1, 16 (1941); *Natl. Advisory Comm. Aeronaut. Tech. Mem.* 1096 (1946).

III. Correlation of Profiles of Total Momentum and Energy Flux in a Nonisothermal Jet Discharging into a Duct

The data in the first paper of this series on the distribution of momentum and energy in nonisothermal air streams mixing in a straight duct were correlated by the methods described in the second paper. Mixing indexes were evaluated and used to correlate profiles of total momentum and stagnation temperature at various sections of the duct.

The understanding and prediction of the processes that take place when a fluid stream mixes turbulently with another of different velocity, composition, or temperature is becoming of ever greater technological importance. This paper studies the mixing of a round jet of heated air, discharged into a straight, cylindrical duct, with a freely induced ambient air stream. The *mixing index*, a quantity which is a measure of the degree to which the two streams remain unmixed at any given section of flow, was computed from the experimental data presented in the first paper of this series and was correlated theoretically by methods described in the second. From this correlation, empirical spreading coefficients were evaluated, and these were used to predict distribution of energy and momentum flux (temperature and velocity distribution) at various sections.

APPARATUS AND PROCEDURE

The aparatus and procedure employed in the experimental work, the results of which are correlated here, have been described in detail in Part I of this series. All the data were obtained at an air-flow rate of 2.07 std. cu. ft./sec., as metered through a calibrated orifice. The air was heated to 222° to 225°F. as measured upstream from the flow nozzle.

RESULTS

Correlation of the Mixing Index for Momentum Transport. On approach to the problem of correlating the momentum data it was clear that account would have to be taken of the scatter in the integral mass flux noted in connection with Figure 14 of Part I. This scatter is amplified in momentum-flux calculations, where the velocity enters as the square, and to an even greater extent in mixing-index calculations, where the velocity enters as the fourth power. It was noticed that the individual points on the velocity profiles did not show much scatter relative to one another and that the continuous curves appeared to be reasonably well defined. It seemed to be more likely that the scatter in the integral mass flux resulted from random changes of 5 to 10% in the flow pattern and induction ratio between successive profile measurements. This effect has been noticed by others(*3*) and is thought to reflect a true physical instability in the system.

It was postulated, however, that the relative shape of a momentum-flux profile, rather than its relative position, is the true measure of the degree of mixing of the two streams, and it seemed reasonable that some method of normalizing the profiles might eliminate the scatter and yield reliable values of the mixing index. Implicit in this is the hypothesis that the mixing is a function only of the geometry of the system and independent of the relative velocities of the initial streams. The manner in which the generalized flux function M was constructed [Equation (7)*] accomplishes the desired normalization as shown by the fact that it leads to an analytical solution [Equations (20) and (25) and discussion] that is independent of the relative velocities, provided the spreading coefficient c is independent also. Although this in general is probably not the case(*2*), minor variations in the flow ratio, as encountered here, should result in only second-order variations in c.

In Part II the necessity of evaluating m_∞ at each cross section because of the unavoidable loss of momentum at the wall of the duct was discussed [Equation (41)]. The

*All equations referred to appear in Part II.